Tutorial ⑤

Limits

Def^n: $\lim_{\vec{x}\to\vec{y}} f(\vec{x}) = \vec{z}$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that: $\|\vec{x}-\vec{y}\| < \delta \Rightarrow \|f(\vec{x}) - z\| < \varepsilon$

Example 1: (1) Using the definition $(\varepsilon-\delta)$ to prove $\lim_{(x,y)\to(1,2)} 2x+3y = 8$

Given $\varepsilon > 0$ we pick $\delta < \varepsilon/5$

$\|x-1\| < \delta$ & $\|y-2\| < \delta$

$$\begin{cases} -\delta < x-1 < \delta \iff -2\delta < 2x-2 < 2\delta \\ -\delta < y-2 < \delta \iff -3\delta < 3y-6 < 3\delta \end{cases}$$

$$-5\delta < 2x+3y-8 < 5\delta < \varepsilon$$

(2) $$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = 0$$

Hint: $|x| \le \sqrt{x^2+y^2+z^2}$

Rough Sketch:

$$\left|\frac{x\cdot y\cdot z}{x^2+y^2+z^2} - 0\right| = \left|\frac{x\ y\ z}{\sqrt{x^2+y^2+z^2}\cdot\sqrt{x^2+y^2+z^2}}\right|$$
$$\le 1\cdot 1\cdot |z|$$
$$\le \sqrt{x^2+y^2+z^2}$$

The formal proof will leave to you to complete.

Continuity

Def^n: Let $A \subset \mathbb{R}^n$ and $P \in A$. If $f: A \to \mathbb{R}^m$ is a function then we say that f is continuous at P if

$$\lim_{Q\to P} f(Q) = f(P)$$

If f is continuous, it is con't at every pt in example 2: Evaluate the limit or show that the limit DNE

$\begin{cases} \text{① sub } (a,b) \text{ into the function} \\ \text{② continuity of the function} \\ \text{③ DNE (finding 2 paths that give different heights} \end{cases}$

(1) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{x\,y\,z}{x^2+y^2+z^2}$

switch to spherical coordinate

$$\begin{cases} x=\rho\cos\theta\sin\varphi \\ y=\rho\sin\theta\sin\varphi \\ z=\rho\cos\varphi \\ x^2+y^2+z^2=\rho^2 \end{cases}$$

$$\lim_{\rho\to0} \frac{\rho^3\cos\theta\sin^2\varphi\sin\theta\cos\varphi}{\rho^2}$$

$$=\lim_{\rho\to0} \rho\cos\theta\sin^2\varphi\sin\theta\cos\varphi$$

$$\le \lim_{\rho\to0} = 0$$

(2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2}{x^2+y^2}$

Along $x=0$:

$$\lim_{(0,y)\to(0,0)} \frac{2\cdot0}{0^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{y^2} = \infty$$

Along $y=0$:

$$\lim_{(x,0)\to(0,0)} \frac{2x^2}{x^2+0^2} = 2$$

limit DNE.

(3) $\lim\limits_{(x,y)\to(0,2)} (1+x)^{y/x} = L.$

$$\log L = \log \lim_{(x,y)\to(0,2)} (1+x)^{y/x} = \lim_{(x,y)\to(0,2)} \frac{y}{x}\log(1+x)$$

$$\le \lim_{(x,y)\to(0,2)} \frac{y}{\cancel{x}}\cdot\cancel{x} = 2$$

$\log L = 2$

$L = e^2$

example 3:

1) Let $f(x,y) = \begin{cases} \dfrac{x^4 - x^3y + 2x^2y^2 - xy^3 + y^4}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$

Find all (x,y) for which $f(x,y)$ is continuous.

Sol'n:

For $(x,y) \neq (0,0)$, $f(x)$ is a rational function.

$$\lim_{(x,y)\to(0,0)} \frac{\cancel{(x^2+y^2)}(x^2-xy+y^2)}{\cancel{x^2+y^2}} = x^2 - xy + y^2 = 0$$

$\therefore f(x,y)$ is con't at $(0,0)$

$\therefore f(x,y)$ is con't for all $(x,y) \in \mathbb{R}^2$

(2) Let $f(x,y) = \begin{cases} \dfrac{3x+xy}{2x^2+(3+y)^2}, & (x,y) \neq (0,-3) \\ -1, & (x,y) = (0,-3) \end{cases}$

Is f con't at $(0,-3)$?

$$\lim_{(x,y)\to(0,-3)} \frac{3x+xy}{2x^2+(3+y)^2}$$

Along $x = 3+y$:

$$\lim_{(x,y)\to(0,-3)} \frac{3(3+y) + (3+y)y}{2(3+y)^2 + (3+y)^2}$$

$$= \lim_{(3+y,y)\to(0,-3)} \frac{\cancel{(3+y)^2}}{3\cancel{(3+y)^2}}$$

$= \frac{1}{3} \neq -1$, f is not continuous at $(0,-3)$.