Tutorial (3)

Cartesian coordinate → polar cood

Cylindrical (x, y, z) (r, θ, z)

Coordinate $x = r\cos\theta$ $y = r\sin\theta$ $z = z$

$$\begin{cases} \cos\theta = \frac{x}{r} \quad \sin\theta = \frac{y}{r} \\ r^2 = x^2 + y^2, \ \tan\theta = \frac{y}{x} \end{cases}$$

$r > 0, 0 \le \theta \le 2\pi$

Spherical Coordinate: $(x, y, z) \to (\rho, \theta, \varphi)$

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\phi \end{cases}$$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

$\tan\theta = \frac{y}{x}$

$\boxed{\rho^2 = x^2 + y^2 + z^2}$

$\cos\phi = \frac{z}{\rho}$

example 1:

(a) Sketch the curve with polar equation $r = 2\cos\theta$

$\cos\theta = \frac{x}{r}$

$r = \frac{2x}{r} \Leftrightarrow r^2 = 2x = x^2 + y^2$

$\Leftrightarrow x^2 + y^2 - 2x = 0$

$\Leftrightarrow (x-1)^2 + y^2 = 1$ ⇒ equation of a circle with center $(1,0)$ and radius 1.

(b) Interpret $\tan\theta = 2$ geometrically.

$\tan\theta = \frac{y}{x} = 2 \Rightarrow y = 2x$.

(c) $\rho\sin\phi = 2$

$$\left.\begin{aligned} x &= \rho\cos\theta\sin\phi = 2\cos\theta \\ y &= \rho\sin\theta\sin\phi = 2\sin\theta. \end{aligned}\right\} \Rightarrow x^2 + y^2 = 2^2.$$

$z = \rho\cos\phi$ $z^2 = \rho^2\cos^2\phi = \rho^2(1 - \sin^2\phi) = \rho^2 - \rho^2\sin^2\phi$

$= \rho^2 - 2^2$

$= x^2+y^2+z^2-2^2 = 2^2+z^2-2^2 = z^2$

Level curves.

Let $f: U \subset \mathbb{R}^2 \to \mathbb{R}$ and let $C \in \mathbb{R}$

The level curve at height C of f is the curve $f(x,y) = C$ in the xy-plane.

The level curves C of height C is $\{(x,y) \in \mathbb{R}^2 \mid f(x,y) = C\}$

example 2: $f(x,y) = \dfrac{x+y}{x^2+y^2-1}$ Sketch and characterize this function.

① domain of the function.

Are there any points where this function is not defined at? If yes. Sketch it out.

$$x^2+y^2-1 \neq 0 \Rightarrow x^2+y^2 \neq 1 \Rightarrow \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \neq 1\}.$$

② Set function $f(x,y) = C$

③ Consider $C = 0$ and $C \neq 0$.

When $C = 0$,

$$\frac{x+y}{x^2+y^2-1} = 0 \iff x+y = 0 \iff y = -x.$$

When $C \neq 0$, (Watch out for features of f)

$$\frac{x+y}{x^2+y^2-1} = C$$ Is it a circle, elipse or a hyperbola?

$$\iff x+y = Cx^2+Cy^2-C$$

$$\iff Cx^2+Cy^2-C-x-y = 0.$$

$$\iff \frac{x+y}{C} = x^2+y^2-1.$$

$$\iff 1+\frac{1}{2C^2} = x^2-\frac{x}{C}+\frac{1}{4C^2}+y^2-\frac{y}{C}+\frac{1}{4C^2}$$

$$\iff 1+\frac{1}{2C^2} = \left(x-\frac{1}{2C}\right)^2+\left(y-\frac{1}{2C}\right)^2.$$

This is a circle centered at $\left(\frac{1}{2c}, \frac{1}{2c}\right)$ with radius $\sqrt{1+\frac{1}{2c^2}}$

④ Pick $c>0$ $\quad c<0$. 5 level curves.

example 3: Sketch a picture showing in the regions in $\mathbb{R}^2$ where the expression is positive or negative. Also indicate when it's at 0 or not defined.

$z = \sin(y^2-x^2)$

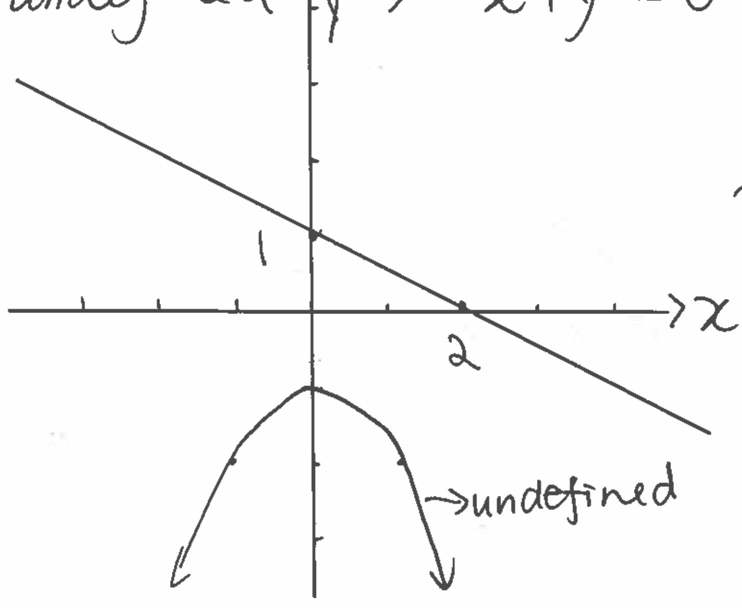
① $c=0 \Rightarrow y^2-x^2 = k\pi,\ k\in\mathbb{Z}$

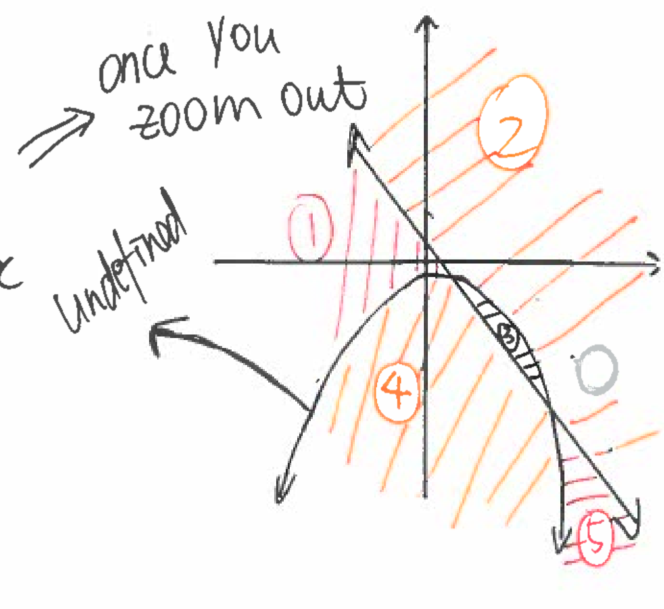
$\Rightarrow (y-x)(y+x) = \begin{cases}\pm k\pi \\ 0\end{cases} \Rightarrow \{\, y=x,\ y=-x$

$c = \frac{x+2y+2}{y+x^2}$.

① $c=0 \Rightarrow x+2y+2=0 \Rightarrow y = 1-\frac{1}{2}x$.

② c undefined $\Rightarrow x^2+y=0 \Rightarrow y=-x^2$





③ Try pairs of points that falls into different regions. In this question we need to find 5 points to label if they are (+)tive or (-)tive.
For region ①, plug in (-100, 0) and find out what z is.

$$z = f(-100, 0) = \frac{-100 + 2 \cdot 0 + 2}{0 + (-100)^2}$$

$$= -0.098$$

Therefore for any points in region ①, z is (-)tive.
Apply the same algorithm for region ②–⑤
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