Tutorial (4)

Lagrange Multiplier

Define $f : \mathbb{R}^n \to \mathbb{R}$. Define a new function $h : \mathbb{R}^{n+1} \to \mathbb{R}$ by $h(\underline{x}, \lambda) = f(\underline{x}) - \lambda(g(\underline{x}) - c)$

Finding the critical point of h will give λ and the constrained CP of f.

example 1: Find the minimum distance between the line $x+y=2$ and the ellipse $x^2+2y^2=2$

$$f(x,y,u,v) = (x-u)^2 + (y-v)^2$$

We want to minimize $f(x,y,u,v)$ subject to 2 constraints

$$\begin{cases} g_1 = x+y-2 = 0 \\ g_2 = u^2+2v^2-2 = 0 \end{cases}$$

$$h(x,y,u,v,\lambda,w) = (x-u)^2 + (y-v)^2 - \lambda(x+y-2) - w(u^2+2v^2-2)$$

$$h_x = 2(x-u) - \lambda = 0$$
$$h_y = 2(y-v) - \lambda = 0$$
$$h_u = -2(x-u) - 2wu = 0$$
$$h_v = -2(y-v) - 4vw = 0$$
$$h_\lambda = -(x+y-2) = 0$$
$$h_w = -(u^2+2v^2-2) = 0$$

6 equations and 6 unknowns!

example 2: Find the points on the curve of intersection of the plane $2x + 2y + z = 2$ and the cylinder $x^2+y^2=4$ which are nearest to and furtherest from the origin.

Solution:

Squared distance is what we are trying to minimizing.

$$f(x,y,z) = (x-0)^2 + (y-0)^2 + (z-0)^2$$

where (x,y,z) can be any point on the curve.

2 constraints are $\begin{cases} g_1(x,y,z) = 2x+2y+z=2 \\ g_2(x,y,z) = x^2+y^2-4 \end{cases}$

$$h(x,y,z,\lambda,\mu) = x^2+y^2+z^2 - \lambda(2x+2y+z-2) - \mu(x^2+y^2-4)$$

Now finding the CP of h:

$$\left.\begin{aligned} h_x &= 2x - 2\lambda - 2\mu x = 0 \\ h_y &= 2y - 2\lambda - 2\mu y = 0 \end{aligned}\right\}$$

$2x-2y-2\mu x+2\mu y=0$

$x-y-\mu(x-y)=0$

$(x-y)(1-\mu)=0$

$x=y$ or $1=\mu$

$$h_z = 2z - \lambda = 0$$

$$h_\lambda = -(2x+2y+z-2) = 0$$

$$h_\mu = -(x^2+y^2-4) = 0$$