2. Let (x,y) be Cartesian coordinates in the plane and let (r,θ) be polar coordinates. Express $\frac{\partial^2 z}{\partial\theta^2}$ in Cartesian coordinates.

$$\frac{\partial z}{\partial\theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial\theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial\theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

$$\frac{\partial\left(\frac{\partial z}{\partial\theta}\right)}{\partial\theta} = \frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial\theta}\right) + \frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial\theta}\right)$$

$$= \underset{①}{\boxed{\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\right)}}\cdot\frac{\partial x}{\partial\theta} + \frac{\partial}{\partial\theta}\left(\frac{\partial x}{\partial\theta}\right)\cdot\frac{\partial z}{\partial x}$$

$$+ \underset{②}{\boxed{\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\right)}}\frac{\partial y}{\partial\theta} + \frac{\partial}{\partial\theta}\left(\frac{\partial y}{\partial\theta}\right)\cdot\frac{\partial z}{\partial y}$$

①

$$\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial x}\cdot\frac{\partial x}{\partial\theta} + \frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial y}\cdot\frac{\partial y}{\partial\theta}$$

$$= \frac{\partial^2 z}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 z}{\partial y^2}\cdot(r\cos\theta)$$

②

$$\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial\left(\frac{\partial z}{\partial y}\right)}{\partial x}\cdot\frac{\partial x}{\partial\theta} + \frac{\partial\left(\frac{\partial z}{\partial y}\right)}{\partial y}\cdot\frac{\partial y}{\partial\theta}$$

$$= \frac{\partial^2 z}{\partial x\partial y}\cdot(-r\sin\theta) + \frac{\partial^2 z}{\partial y^2}\cdot(r\cos\theta)$$

$$\frac{\partial\left(\frac{\partial z}{\partial\theta}\right)}{\partial\theta} = ①\cdot(-r\sin\theta) + (-r\sin\theta)\cdot\frac{\partial z}{\partial x} + ②\cdot(r\cos\theta) + (r\cos\theta)\cdot\frac{\partial z}{\partial y}$$

I will leave the rest to you guys ...

Taylor series with Another Base Point

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2!}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(y-b)^2\right] + \dots$$

→ Linear approx. involves f_x, f_y

→ Quadratic approx. involves f_{xx}, f_{xy}, f_{yy}

Example 2: We find the 2^{nd} order Taylor expansion of $f(x,y) = \sqrt{1+4x^2+y^2}$ about $(1,2)$ and use it to compute approximately $f(1.1, 2.05)$.

$f(x,y) = \sqrt{1+4x^2+y^2}$ $\qquad f(1,2) = 3$

$f_x(x,y) = \dfrac{4x}{\sqrt{1+4x^2+y^2}}$ $\qquad f_x(1,2) = \frac{4}{3}$

$f_y(x,y) = \dfrac{y}{\sqrt{1+4x^2+y^2}}$ $\qquad f_y(1,2) = \frac{2}{3}$

$f_{xx}(x,y) = \dfrac{4}{\sqrt{1+4x^2+y^2}} - \dfrac{16x^2}{\sqrt{1+4x^2+y^2}}$ $\qquad f_{xx}(1,2) = \frac{20}{27}$

$f_{xy}(1,2) = -\frac{8}{27}$

$f_{yy}(1,2) = \frac{5}{27}$

$$f(x,y) = 3 + \frac{4}{3}(x-a) + \frac{2}{3}(y-b) + \frac{10}{27}(x-a)^2 - \frac{8}{27}(x-a)(y-b) + \frac{5}{27}(y-b)^2 + E$$

$(x-a) = 0.1 \qquad y-b = 0.05$

$$f(1.1, 2.05) \approx 3 + \frac{4}{3}(0.1) + \frac{2}{3}(0.05) + \frac{10}{27}(0.01) - \frac{8}{27}(0.005) + \frac{5}{27}(0.0025) = 3.1691.$$

example 3. Find and classify all critical points.

$\sin(x) + \sin(y) + \sin(x+y)$ on $(0,\pi)\times(0,\pi)$

$f_x = \cos(x) + \cos(x+y)$

$f_y = \cos(y) + \cos(x+y)$

$f_x = 0,\ f_y = 0$

$$\begin{cases} \cos x + \cos(x+y) = 0 \Rightarrow \cos x = -\cos(x+y) \\ \cos y + \cos(x+y) = 0 \Rightarrow \cos y = -\cos(x+y) \end{cases}$$

$\therefore \cos y = \cos x$

$\therefore y = x \qquad 0 < x, y < \pi$

$\cos(x) + \cos(2x) = 0$

$\cos(x) + 2\cos^2 x - 1 = 0$

$$\cos(x) = \frac{-1 \pm \sqrt{1+8}}{4}$$

$\cos(x) = -1$ or $\cos(x) = \frac{1}{2}$

$\uparrow$ not in the range

$\uparrow$ Solution $x = \frac{\pi}{3}$

CP: $(\frac{\pi}{3}, \frac{\pi}{3})$

$$Hf = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \begin{vmatrix} -\sin x - \sin(x+y) & -\sin(x+y) \\ -\sin(x+y) & -\sin y - \sin(x+y) \end{vmatrix}$$

$$Hf\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \begin{vmatrix} -\sqrt{3} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\sqrt{3} \end{vmatrix}$$

$$\det(A_1) = -\sqrt{3}$$

$$\det(A_2) = \frac{9}{4} > 0$$

$- \; + \; - \; + \; \cdots$ local max.