Tutorial (6)

Ex1. Find the following derivatives

(a) $f: \mathbb{R}^4 \to \mathbb{R}^2$, $f(x_1, x_2, x_3, x_4) = (x_1x_2x_3,\ x_4\tan(x_1x_3))$

Sol'n:

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \end{pmatrix}$$

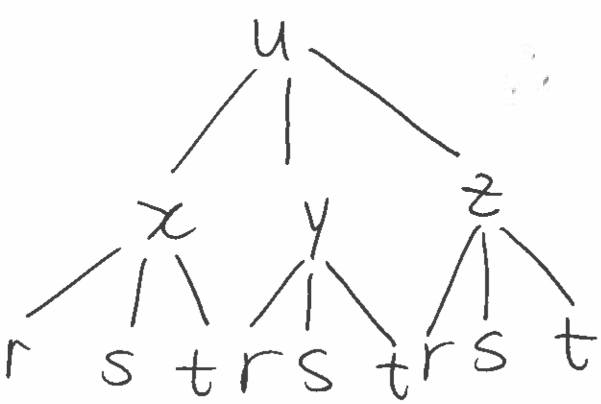
$$= \begin{pmatrix} x_2x_3 & x_1x_3 & x_1x_2 & 0 \\ x_4x_3\sec^2(x_1x_3) & 0 & x_4x_1\sec^2(x_1x_3) & \tan(x_1x_3) \end{pmatrix}$$

(b) If $u = x^4y + y^2z^3$, $y = rse^{-t}$, $x = rse^{t}$, $z = r^2 s\sin t$

Find $\frac{\partial u}{\partial s}$ when $r=2$, $s=1$, $t=0$.

Sol'n:

Note: Build a tree structure to see the relationship if you are not familiar with partial derivatives.



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$\frac{\partial u}{\partial s} = 4x^3y \cdot re^{t} + (x^4 + 2yz^3) \cdot re^{-t} + 3z^2y^2 \cdot r^2 \sin t$$

$$= 4x^3y \cdot (2) + (x^4 + 2yz^3) \cdot (2) + 3z^2y^2 \cdot (0)$$

$$= 4(rse^{t})^3 \cdot (rse^{-t}) \cdot 2 + [(rse^{t})^4 + 2 \cdot (rse^{-t}) \cdot (r^2 s \sin t)] \cdot 2$$

$$= 4 \cdot 2^3 \cdot 2 \cdot 2 + [2^4] \cdot 2 = 160$$

ex2. Evaluate $\frac{\partial f}{\partial x}$ at the point $\vec{a}$.

(a) $f(x) = \|(x, y)\|$, $\vec{a} = (-1, 2)$

Sol'n:

$$f(x) = \sqrt{x^2 + y^2}$$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \left.\frac{\partial f}{\partial x}\right|_{(-1,2)} = \frac{-1}{\sqrt{5}} = -\frac{1}{\sqrt{5}}$$

(b) $f(x) = \begin{cases} \frac{3xy + 5y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ at $\vec{a} = (0,0)$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0}{h^3}$$

$$= 0$$

ex 3.

$f : \mathbb{R}^3 \to \mathbb{R}^3 \quad f(x,y,z) = (xy,\ z^3,\ y^2z)$

$g : \mathbb{R}^3 \to \mathbb{R}^2 \quad g(x,y,z) = (x^2+y^2,\ xyz)$

Find Df and Dg. Use chain rule to find $D(g \circ f)$

Sol'n:

Basically need to find : Df, Dg, Dg(f(a)), and Dg(f(a)) · Df(a)

$$Df = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \end{pmatrix} = \begin{pmatrix} y & x & 0 \\ 0 & 0 & 3z^2 \\ 0 & 2yz & y^2 \end{pmatrix}$$

$$Dg = \begin{pmatrix} \nabla g_1 \\ \nabla g_2 \end{pmatrix} = \begin{pmatrix} 2x & 2y & 0 \\ yz & xz & xy \end{pmatrix}$$

$$Dg(f(a)) = \begin{pmatrix} 2xy & 2z^3 & 0 \\ z^3y^2z & xy^3z & xyz^3 \end{pmatrix}$$

$D(g \circ f) =$

$$Dg(f(a)) \cdot Df(a) = \begin{pmatrix} 2xy & 2z^3 & 0 \\ z^4y^2 & xy^3z & xyz^3 \end{pmatrix} \begin{pmatrix} y & x & 0 \\ 0 & 0 & 3z^2 \\ 0 & 2yz & y^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2xy^2 & 2x^2y & 6z^5 \\ z^4y^3 & xz^4y+2xy^2z^4 & 3xy^3z^3+xy^3z^3 \end{pmatrix}$$

[If I made a calculation mistake somewhere in ex3, please let me know ☺]